ELSEVIER Linear Algebra and its Applications 313 (2000) 203–206

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

A test for copositive matrices

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Received 23 October 1999; accepted 4 April 2000

Submitted by R.A. Brualdi

Abstract

The paper presents necessary and sufficient conditions that a symmetric matrix be copositive or strictly copositive. The conditions are given in terms of the eigenvalues and eigenvectors of the principal submatrices of the given matrix. © 2000 Elsevier Science Inc. All rights reserved.

AMS classification: 15A18; 15A48; 15A63; 49N10; 65K99

Keywords: Copositive matrices; Eigenvalues; Optimization

A real symmetric matrix A of order n is said to be copositive if $x^{\mathrm{T}}Ax \geqslant 0$ for $x \geqslant 0$. It is termed strictly copositive if it is copositive and equality holds only for $x = 0$. There is an extensive literature on such matrices; see for example [1–5,7–10] and references cited therein.

In this paper, we give necessary and sufficient conditions for these two classes of matrices. The conditions can be derived fairly easily from Theorems 3.1 and 3.2 of [2]. However, the statement of the conditions and the very short direct proofs should be of interest.

In the proofs we denote by $Q(x)$ the quadratic form $x^{\mathrm{T}}Ax$, by $\|x\|$ the Euclidean norm of x, and by $\mathbb{R}^n_{\mathrm{pos}}$ the set of all x in $\mathbb{R}^n$ such that $x \geqslant 0$, $x \neq 0$.

Theorem 1. *Let A be a symmetric matrix of order n. Then A is strictly copositive if and only if every principal submatrix B of A has no eigenvector $v > 0$ with associated eigenvalue $\lambda \leqslant 0$.*

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PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 1 3 8 - 5

Proof. Let the principal submatrices of A have the property stated in the theorem and let $Q(x_0) \leqslant 0$ for some x_0 in $\mathbb{R}^n_{\text{pos}}$. Some, but not all, components of x_0 may be 0. For proper numbering, we can assume that $x_0 = (a_1, \dots, a_m, 0, \dots, 0)^{\mathrm{T}}$, where $1 \leqslant m \leqslant n$, and $a_i > 0$ for $i = 1, \dots, m$. The vector x_0 cannot be the unique one at which Q has a nonpositive value; among all such vectors, we can choose one at which m has its least value, and we assume that our x_0 has this property.

We consider the case $m > 1$. Let y be an arbitrary vector of $\mathbb{R}^m$ and let

$$Q_0(y) = Q(y_1, \dots, y_m, 0, \dots, 0).$$

Let $y_0 = (a_1, \dots, a_m)^{\mathrm{T}}$. We consider the function $Q_0(y)$ restricted to the set E: $\|y\| = 1$, $y \geqslant 0$. By multiplying y_0 by a positive scalar, we can assure that y_0 is in E. The vector y_0 must then be in the relative interior of E, since a vector in the relative boundary of E would have less than m nonzero components. Thus, the function $Q_0(y)$ on E has positive values on the relative boundary of E and has a nonpositive value at one point in the relative interior. Accordingly, the function must have a negative or zero absolute minimum at some vector v in the relative interior, so that $v > 0$. But the vector v would be an eigenvector of a principal submatrix of A with a negative or a zero associated eigenvalue. By hypothesis, there can be no such vector. Accordingly, $Q(x) > 0$ in $\mathbb{R}^n_{\text{pos}}$.

For the case $m = 1$, the principal submatrix $[a_{11}]$ would have the eigenvector $v = (1)$ with a nonpositive eigenvalue and the same conclusion follows.

Conversely, let $Q(x) > 0$ in $\mathbb{R}^n_{\text{pos}}$ and let some principal submatrix B of A have an eigenvector $v > 0$ with a nonpositive associated eigenvalue. We can assume that B is obtained from A by deleting the rows and columns following the mth, and write $v = (a_1, \dots, a_m)^{\mathrm{T}}$, $x_0 = (a_1, \dots, a_m, 0, \dots, 0)^{\mathrm{T}}$. Then $Q(x_0) = v^{\mathrm{T}} B v \leqslant 0$, contrary to hypothesis. Therefore, no principal submatrix of A can have an eigenvector $v > 0$ with a negative or a zero associated eigenvalue.

Thus, Theorem 1 is proved. □

Theorem 2. *Let A be a symmetric matrix of order n. Then A is copositive if and only if every principal submatrix B of A has no eigenvector $v > 0$ with associated eigenvalue $\lambda < 0$.*

The proof parallels that of Theorem 1 with appropriate changes in the inequalities.

Example. We consider the matrix A of the quadratic form

$$Q(x) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 + 6x_2x_3.$$

The principal matrices of order 1 are the diagonal elements, all equal to 2, and these matrices have only the positive eigenvalue 2. Of those of order 2, two have the eigenvalues 1 and 3, and the remaining one has the eigenvalues 5 and -1, with eigenvectors $c(1, -1)^{\mathrm{T}}$ associated to the eigenvalue -1. The principal submatrix of order 3 is A itself, which has the eigenvalues 5.5616, 1.4384 and -1. The eigenvectors associated with the eigenvalue -1 are multiples of $(0, -1, 1)^{\mathrm{T}}$. In no case is there a nonpositive eigenvalue with associated eigenvector $v > 0$. Accordingly, A is strictly copositive.

For this example, it is not difficult to verify directly that $Q(x) > 0$ for $x \geqslant 0$, $x \neq 0$. The matrix A is not positive definite and some entries are negative; if it were positive definite or all entries were positive, then of course it would have to be strictly copositive.

Geometric interpretation: The idea behind the proofs arose from simple geometric reasoning. We illustrate this by considering the case $n = 3$ and a matrix A, which has one negative eigenvalue and two positive eigenvalues. After a rotation of axes, the set on which $Q(x) \leqslant 0$ can be represented by an inequality

$$ax_1^2 + bx_2^2 - cx_3^2 \leqslant 0,$$

where a, b, and c are positive. The set is bounded by the two nappes of an elliptical cone; we denote by E_1, and E_2 the solids bounded by the two nappes. If we reverse the rotation, one of the solids, say E_1, may intersect the first octant at an interior point P of E_1; this occurs precisely when A is not copositive. If this occurs, but no such point P exists on a coordinate plane, then $Q(x)$ is nonnegative on the boundary of the first octant but takes on negative values at some interior points of the first octant; this leads to a negative eigenvalue of A with positive eigenvector; if such a point P exists on a coordinate plane but not on a coordinate axis, then similar reasoning leads to a negative eigenvalue with a positive eigenvector of a principal submatrix of A having order 2; if such a point P exists on a coordinate axis, then we have a similar statement for order 1.

Remark. Copositive matrices occur in optimization theory. See for example [3, Chapter 3], [4,5], [6, p. 133] and [9, Chapter 2]. The tests of Theorems 1 and 2 are of little practical value for such applications when the matrix has a large order, since a matrix of order n has $2^n - 1$ principal submatrices.

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